PRINCIPLES OF ANALYSIS
TOPIC VII: CONTINUITY
*** DRAFT ***

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ABSTRACT. We discuss continuous functions, and show that the continuous image of a compact set is compact and that the continuous image of a connected set is connected. Together with the Heine-Borel Theorem, this implies the Intermediate Value Theorem.

1. CONTINUITY

Definition 1. Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ and $a \in D$. We say that f is *continuous at a* if

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \ni \; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

Let $A \subset D$. We say that f is *continuous on A* if f is continuous at a for every $a \in A$, We say that f is *continuous* if f is continuous on its entire domain.

Proposition 1. *Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ and $a \in D$. Then f is continuous at a if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in D which converges to a, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(a)$.*

Proposition 2. *Let $f : E \to \mathbb{R}$ be a continuous function, and let $A \subset E$ be connected. Then $f(A)$ is connected.*

Proof. It suffices to show that if $f(A)$ is disconnected, then A is disconnected. Thus assume that $f(A)$ is disconnected, and let V_1 and V_2 be open subsets of $\mathbb{R}$ such that $f(A) \cap V_1 \neq \varnothing$, $f(A) \cap V_2 \neq \varnothing$, but $f(A) \subset (V_1 \cup V_2)$. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Then $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, but $A \subset (U_1 \cup U_2)$. Moreover, since f is continuous, U_1 and U_2 are open. Thus, A is disconnected. □

Proposition 3. *Let $f : E \to \mathbb{R}$ be a continuous function, and let $A \subset E$ be a compact set. Then $f(A)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(A)$. Define

$$\mathcal{B} = \{f^{-1}(V) \mid V \in \mathcal{C}\}.$$

Since A is compact, there exists a finite subset of $\mathcal{B}$, say $\mathcal{U} = \{U_1, \dots, U_n\}$, such that $A \subset \cup_{i=1}^{n} U_i$. Each U_i is the preimage of an open subset, say $U_i = f^{-1}(V_i)$. Then $f(U_i) \subset V_i$, and

$$f(A) \subset f(\cup_{i=1}^{n} U_i) = \cup_{i=1}^{n} f(U_i) \subset \cup_{i=1}^{n} V_i;$$

now $\mathcal{V} = \{V_1, \dots, V_n\}$ is a finite subcover of $\mathcal{C}$. This shows that $f(A)$ is compact. □

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